Name: ..……………………………………………………………………………….

Student Number: ………………………………………………………………………..

Test 5 on WPPH16001.2019-2020
"Electricity and Magnetism"

Content: 7 pages (including this cover page)

Friday May 22 2020; **online**, 15:00-17:30

- Write your full name and student number on **each** page you use
- Read the questions carefully. Read them one more time after having answered them.
- Compose your answers is such a way that it is well indicated which (sub)question they address
- Upload the answer to each question as a **separate pdf file**
- Do not use a red pen (it's used for grading)
- Griffiths' textbook, lecture notes and **your** tutorial notes are allowed. The internet, mobile phones, consulting and other teamwork are not allowed (and considered as cheating)

Exam drafted by (name first examiner) Maxim S. Pchenitchnikov
Exam reviewed by (name second examiner) Steven Hoekstra

For administrative purposes; do NOT fill the table

The weighting of the questions:

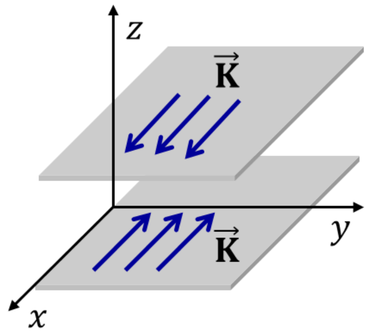
	Maximum points	Average points scored
Question 1	15	9
Question 2	10	6.8
Question 3	10	5
Question 4	10	8.8
Total	**45**	**28.7**

Grade = 1 + 9 x (score/max score).

Averaged grade: 6.75

Question 1. (15 points)

Consider two infinite planes, parallel to the xy plane, some distance apart. The top plane carries a surface current density $\vec{\mathbf{K}} = K\,\hat{\mathbf{x}}$ and the bottom plane carries the same surface current in the opposite direction.



1. Show that the magnetic field between the plates is given by: $\vec{\mathbf{B}} = \mu_0 K\,\hat{\mathbf{y}}$ (2 points)

2. Show that the Maxwell stress tensor $\overleftrightarrow{\mathbf{T}}$, expressed in the matrix form is given as:

$$\overleftrightarrow{\mathbf{T}} = \frac{\mu_0 K^2}{2}\begin{pmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{(5 points)}$$

3. Calculate the electromagnetic pressure on the top plate by the bottom plate using the Maxwell stress tensor. (5 points)

4. Now calculate the electromagnetic force per unit area on the top plate by the bottom plate using Lorentz' force law. (2 points)

5. Do you expect to obtain the same result in (4) as in (3)? Explain your answer. (1 point)

Model answers **(15 points)**

1. Calculate the field for one plate first (btw, you did this a few times earlier). Draw a rectangular ampèrian loop with the length of l in the yz plane as we did considering the boundary conditions (Lecture 22). Use Ampère-Maxwell's law with the time-derivative term nullified (the currents are stationary):

$$\oint_{\mathcal{P}} \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{l}} = \mu_0 I_{fenc} \qquad \text{1/2 point}$$

As the $\vec{\mathbf{B}}$ field is in the y direction,

$$2Bl = \mu_0 K l$$

$$\vec{\mathbf{B}} = \frac{\mu_0 K}{2}\hat{\mathbf{y}} \qquad \text{1/2 point}$$

For two plates, the fields add up:

$$\vec{\mathbf{B}} = \mu_0 K\,\hat{\mathbf{y}} \qquad \text{1 point}$$

Note that by the virtue of the same argument, $\vec{\boldsymbol{B}}$ out of space between the plates is zero. If it helps: the setting begins to resemble a plane capacitor (Problem 8.7), with the electric field substituted by magnetic field. Therefore, the further solution goes along similar lines – or simply use a brute force approach.

2. The Maxwell stress tensor (no electric field) is given by:

$$T_{ij} \equiv \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2\right)$$

All off diagonal components of $\overleftrightarrow{\mathbf{T}}$ are zero, because both B_x and B_z are zero:

$$T_{ij} = 0 \text{ if } i \neq j \qquad \text{1 point}$$

The diagonal components are

$$T_{xx} \equiv \frac{1}{\mu_0}\left(B_x B_x - \frac{1}{2}B_y^2\right) = -\frac{\mu_0 K^2}{2}$$ 1 point

$$T_{yy} \equiv \frac{1}{\mu_0}\left(B_y B_y - \frac{1}{2}B_y^2\right) = +\frac{\mu_0 K^2}{2}$$ 1 point

$$T_{zz} \equiv \frac{1}{\mu_0}\left(B_z B_z - \frac{1}{2}B_y^2\right) = -\frac{\mu_0 K^2}{2}$$ 1 point

$$\overleftrightarrow{\mathbf{T}} = \frac{\mu_0 K^2}{2}\begin{pmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$ 1 point

only if all previous steps are correct (otherwise it's a copy – paste from the question)

3. The force as calculated from the Maxwell stress tensor is given by:

$$\vec{\mathbf{F}} = \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\vec{\mathbf{a}}$$

The area element is $d\vec{\mathbf{a}} = -dx\, dy\, \hat{\mathbf{z}}$ because the force is on the bottom side of the top plate (outside the field is zero). 1 point

$$\vec{\mathbf{F}} = \frac{\mu_0 K^2}{2}\oint_{\mathcal{S}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ -dx\, dy \end{pmatrix}$$ 1 point

$$= \frac{\mu_0 K^2}{2}\oint_{\mathcal{S}} dx\, dy\, \hat{\mathbf{z}}$$ 1 point

$$= \frac{\mu_0 K^2}{2} A\, \hat{\mathbf{z}}$$ 1 point

where A is the area of the plate.

The electromagnetic pressure is force per unit area

$$\vec{\mathbf{f}} = \frac{\vec{\mathbf{F}}}{A} = \frac{\mu_0 K^2}{2}\, \hat{\mathbf{z}}$$ 1 point

Note that conductors carrying anti-parallel currents repel each other so that force on top plate should be in +z direction as we derived.

4. The force due to the field of the lower plate is given by:

$$\vec{\mathbf{f}} = \rho\vec{\mathbf{E}} + \vec{\mathbf{J}}_{Upper} \times \vec{\mathbf{B}}_{Lower}$$

$$= \vec{\mathbf{J}}_{Upper} \times \vec{\mathbf{B}}_{Lower} = K\hat{\mathbf{x}} \times \frac{\mu_0 K}{2}\hat{\mathbf{y}}$$ 1 point (0.5 pts if B is for both plates)

$$= \frac{\mu_0 K^2}{2}\, \hat{\mathbf{z}}$$ 1 point (0.5 pts if direction is wrong)

Exactly as in (3).

<u>Alternative solution:</u>

$$\vec{\mathbf{F}} = \int d\vec{\mathbf{I}}_{Upper} \times \vec{\mathbf{B}}_{Lower}$$

$$d\vec{\mathbf{I}}_{Upper} = K\, dx\, dy\, \hat{\mathbf{x}}$$

$$\vec{\mathbf{F}} = \oint_S K\,dx\,dy\,\hat{\mathbf{x}} \times \vec{\mathbf{B}}_{Lower} = K\frac{\mu_0 K}{2} A\,\hat{\mathbf{z}}$$ 1 point

Force per unit area:

$$\vec{\mathbf{f}} = \frac{\vec{\mathbf{F}}}{A} = \frac{\mu_0 K^2}{2}\,\hat{\mathbf{z}}$$ 1 point

5. The result must be the same because the Maxwell stress tensor formalism is simply a more convenient way of expression the Lorentz force, exactly what we did at the lectures. (1 point)

Typical mistakes:
Q1.1: Using the integral form of Ampère's law without specifying a loop and thereby arriving at a wrong answer. (e.g. not considering each plate as separate)
Q1.1: Using the boundary condition approach incorrectly.
Q1.3: wrong direction of the surface vector or no vector representation at all (very common)
Q1.3: Dividing the tensor by two, because we are only interested in one plate.
Q1.4: Saying that **K** = - K **z**^ in the top plate.
Q1.4: Not knowing how to properly exchange the quantities q, σ, A, K, da, dq, v in whatever formulation of the Lorentz force.

Question 2. (10 points)

A plane monochromatic electromagnetic wave of frequency ν, polarized in the positive z-axis direction moves in the positive y-axis direction. The amplitude of the electric field is E_0, and the start of time is chosen so that at $t = 0$, the electric field has a value $E_0/2$ at the origin. In the answers, use *only the variables provided above*, the speed of light c and permittivity ϵ_0.

1. Write the electric field of the wave. (2 points)

2. Find the associated magnetic field. (2 points)

3. Find the Poynting vector associated with this wave. (2 points)

4. Show by direct averaging of the Poynting vector that intensity of the wave is $I = \frac{1}{2}c\epsilon_0 E_0^2$ (2 points)

5. Find the expression for the momentum density stored in this wave (2 points)

<u>Model answers</u> (10 points)
(Where applicable, 1 point for correct magnitude and 1 point for correct direction)

1. $\vec{\mathbf{E}}(y,t) = E_0\cos\left(\frac{2\pi\nu}{c}y - 2\pi\nu t + \frac{\pi}{3}\right)\hat{\mathbf{z}} = E_0\cos\left[\frac{2\pi\nu}{c}(y - ct) + \frac{\pi}{3}\right]\hat{\mathbf{z}}$ (2 points)

2. $\vec{\mathbf{B}}(y,t) = \frac{E_0}{c}\cos\left(\frac{2\pi\nu}{c}y - 2\pi\nu t + \frac{\pi}{3}\right)\hat{\mathbf{x}} = \frac{E_0}{c}\cos\left[\frac{2\pi\nu}{c}(y - ct) + \frac{\pi}{3}\right]\hat{\mathbf{x}}$ (2 points)

3. $\vec{\mathbf{S}} = \frac{1}{\mu_0}(\vec{\mathbf{E}} \times \vec{\mathbf{B}}) = \frac{E_0^2}{\mu_0 c}\cos^2\left[\frac{2\pi\nu}{c}(y - ct) + \frac{\pi}{3}\right]\hat{\mathbf{y}} = \epsilon_0 c E_0^2\cos^2\left[\frac{2\pi\nu}{c}(y - ct) + \frac{\pi}{3}\right]\hat{\mathbf{y}}$
(2 points)

4. $I \equiv \langle S\rangle_t = c\epsilon_0 E_0^2\left\langle\cos^2\left[\frac{2\pi\nu}{c}(y - ct) + \frac{\pi}{3}\right]\right\rangle_t = \frac{1}{2}c\epsilon_0 E_0^2$ (2 points)

5. $\vec{\mathbf{g}} = \epsilon_0\mu_0\vec{\mathbf{S}} = \epsilon_0^2\mu_0 c E_0^2\cos^2\left[\frac{2\pi\nu}{c}(y - ct) + \frac{\pi}{3}\right]\hat{\mathbf{y}} = \frac{\epsilon_0}{c}E_0^2\cos^2\left[\frac{2\pi\nu}{c}(y - ct) + \frac{\pi}{3}\right]\hat{\mathbf{y}}$
(2 points)

Typical mistakes:
General: Use of variables that are not allowed (e.g. k, ω, μ_0)

Confusion between ν and ω (frequency vs angular frequency)
Instead of a phase difference, the amplitude is scaled by 1/2 or even $E_0/2$ was added to the wave, resulting in cross terms (when squaring) for the rest of the question.
Stating that $\cos^2$ averages to 1/2 without explaining it while the question was formulated as “show by direct averaging”

Question 3. (10 points)

Consider an ideal parallel-plate capacitor with the separation between the plates of d. The capacitor is filled with an isotropic linear dielectric material which frequency-dependent electric susceptibility can be approximated as $\chi_e = \chi_0/(1+\omega^2\tau^2)$, where χ_0 and τ are constants. The capacitor is driven by a voltage $V = V_0 \cos \omega t$.

1. Show that the displacement current density is

$J_d = -\dfrac{\epsilon_0 V_0}{d}\omega\left(1+\dfrac{\chi_0}{1+\omega^2\tau^2}\right)\sin\omega t$ (3 points)

2. Find the amplitude J_{d_0} of the displacement current density. (1 point)

3. Find the difference between the *amplitudes* of displacement current densities of the filled and empty capacitors. (2 points)

4. At which frequency is the difference maximal? (3 points)

5. What is the value of the electric susceptibility at this frequency with respect to χ_0? (1 points)

NB.This method can be used for measuring χ_0 and τ

Model answers **(10 points)**

1. $J_d = \dfrac{\partial D}{\partial t}$

$E = \dfrac{V}{d} = \dfrac{V_0\cos\omega t}{d}$ (1 point)

$D = \epsilon E = \epsilon_0(1+\chi_e)E = \epsilon_0\left(1+\dfrac{\chi_0}{1+\omega^2\tau^2}\right)\dfrac{V_0\cos\omega t}{d}$ (1 point)

$J_d = -\dfrac{\epsilon_0 V_0}{d}\omega\left(1+\dfrac{\chi_0}{1+\omega^2\tau^2}\right)\sin\omega t$ (1 point)

2. $J_d = J_{d_0}\sin\omega t$ where J_{d_0} is the amplitude of the displacement current

$J_{d_0} = -\dfrac{\epsilon_0 V_0}{d}\omega\left(1+\dfrac{\chi_0}{1+\omega^2\tau^2}\right)$ (1 point)

3. The difference:

$\Delta J_{d_0} = -\dfrac{\epsilon_0 V_0}{d}\omega\left(1+\dfrac{\chi_0}{1+\omega^2\tau^2}\right) + \dfrac{\epsilon_0 V_0}{d}\omega = -\dfrac{\epsilon_0 V_0}{d}\omega\dfrac{\chi_0}{1+\omega^2\tau^2}$ (2 points)

4. $\dfrac{\partial \Delta J_{d_0}}{\partial\omega} = (\ldots)\dfrac{1+\omega^2\tau^2-\omega 2\omega\tau^2}{(1+\omega^2\tau^2)^2} = 0$ (2 points)

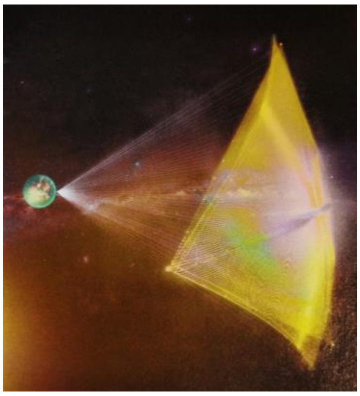
$1-\omega_{max}^2\tau^2 = 0;\ \omega_{max} = \dfrac{1}{\tau}$ (1 point)

5. $\chi_e = \dfrac{\chi_0}{1+\omega^2\tau^2} = \dfrac{\chi_0}{2}$ (1 point)
i.e. a half of the zero-frequency value.

Typical mistakes:

3.1: using $J_d = \epsilon_0 \; dE/dt$ instead of $J_d = \frac{\partial D}{\partial t}$. Of course, the final answer should be patched to have ϵ given in the question – but this is still a mistake.
3.2: confusing the amplitude with the part before the bracket.
3.3: confusing "difference" (i.e. subtraction two values) with ratio (i.e. division of two values)
3.4: Let $\omega = 0$ to be the frequency of the maximal difference.
3.4: Confusion frequency and angular frequency (not a big mistake but nevertheless)

Question 4. (10 points)



In the Starshot initiative, which we discussed at lectures, the idea is to accelerate in the outer space a spacecraft equipped with a lightsail by a laser beam to speeds of one-fifth of the speed of light.

Let's take optimistic numbers: the mass of the whole spacecraft is $m = 1\ g$, the sail area is $A = 1\ m^2$, the sail is perfectly reflective (R =1) and oriented normal to the laser beam, no gravity, no friction. You have a $W = 1\ MW$ ($10^6\ W$) laser (this is a *very* powerful laser!) which could be focused in such a way that the laser beam homogeneously covers the whole sail.

Calculate in numbers (and don't forget about the units!):

1. The radiation pressure exerted by the laser beam upon the lightsail (2 points)

2. The force applied to the lightsail (2 points)

3. How long does it take to reach the speed $v = 0.2c$ from the rest state (without considering relativistic effects)? Provide the final answer in values that are easily understood – i.e. instead of e.g. $10^3\ s$, give ~17 min. (2 points)

4. How many lasers should be used to provide the same speed in 5 minutes? (2 points)

5. What fraction of the world's electricity generating capacity of ~$5{\cdot}10^{12}$ W is required to power all these lasers? Assume electricity-to-light conversion efficiency in the laser as 10%. (2 points)

Model answers (10 points)

(1 point for correct approach, and 1 point for the correct numerical value)

1. $p = 2\frac{I}{c} = 2\frac{W}{Ac} = 2\frac{10^6}{1 \cdot 3 \cdot 10^8} = \frac{2}{3} \cdot 10^{-2}\ \mathrm{N} \cdot \mathrm{m}^{-2}$ (2 points)

2. $F = pA = \frac{2}{3} \cdot 10^{-2}\ \mathrm{N}$ (2 points)

(It has been a typo in #3-#5 answers; corrected)

3. $t = \frac{v}{a} = \frac{0.2c \cdot m}{F} = \frac{0.2 \cdot 3 \cdot 10^8 \cdot 10^{-3}}{2/3 \cdot 10^{-2}} = 9 \cdot 10^6\ s$, or 104 days (2 points)

4. We need enhancement by a factor of $9 \cdot 10^6/300$, or 30000 lasers. (2 points)

5. The power produced by all lasers is $3\text{x}10^{10}$ W which requires 6% of the world's electricity generating capacity. (2 points)

Typical mistakes:

4.1: Most common mistake is omitting the factor of 2 for the pressure
4.3: An attempt to use kinetic energy resulted in a factor of 10 error.

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Maxim Pchenitchnikov
May 17 2020

Steven Hoekstra